E2.5 Signals & Linear Systems

Tutorial Sheet 3 - Solutions.

1. (a) $u(t) * u(t)$

$$= \int_0^t u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_0^t d\tau = \tau\Big|_0^t = t \qquad \text{for } t \geq 0$$

$$= 0 \qquad \text{for } t < 0$$

$$\therefore\ u(t) * u(t) = t\, u(t)$$

(b) $e^{-at} u(t) * e^{-bt} u(t)$

$$= \int_0^t e^{-a\tau} e^{-b(t-\tau)} d\tau = e^{-bt} \int_0^t e^{(b-a)\tau} d\tau$$

$$= \frac{e^{-bt}}{b-a} e^{(b-a)\tau}\Big|_0^t = \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right]$$

$$= \frac{e^{-at} - e^{-bt}}{a-b}.$$

Because both functions are casual, their convolution is zero for $t<0$.

$$\therefore\ y(t) = e^{-at} u(t) * e^{-b(t)} u(t)$$

$$= \left(\frac{e^{-at} - e^{-bt}}{a-b}\right) u(t).$$

(c) Both functions are causal, ∴

$$t\,u(t) * u(t) = \int_0^t \tau\, u(\tau)\, u(t-\tau)\, d\tau$$

For the range of integration, $0 \le \tau \le t$,

and $u(\tau)$ at $\tau = 0$ is $u(0) = 1$, at $\tau = t$ $u(t) = 1$

and $u(t-\tau)$ at $\tau = 0$ is $u(t) = 1$, at $\tau = t$, $u(0) = 1$

∴ $u(\tau) = u(t-\tau) = 1$, and

$$t\,u(t) * u(t) = \int_0^t \tau\, d\tau$$

$$= \frac{t^2}{2} \quad \text{for } t \geq 0$$

$$= 0 \quad \text{for } t < 0$$

∴ $\underline{y(t) = \frac{1}{2} t^2 u(t)}$.

2. a) $y(t) = \sin t\, u(t) * u(t)$

$$= \left[\int_0^t \sin\tau\, u(\tau)\, u(t-\tau)\, d\tau\right] u(t)$$

$$= \left[\int_0^t \sin\tau\, d\tau\right] u(t) \qquad \begin{matrix} u(\tau) = 1 \\ u(t-\tau) = 1 \end{matrix}$$

$$= (1 - \cos t)\; u(t).$$

b) Similarily,

$$y(t) = \left[\int_0^t \cos\tau\, d\tau\right] u(t) = \sin t\; u(t)$$

4. The key is to realise that

$$f(t) = u(t) - u(t-1).$$

$$\therefore \; e^{-t}u(t) * u(t) \Longleftrightarrow \underbrace{(1-e^{-t})u(t)}_{z(t)} \qquad Ⓐ$$

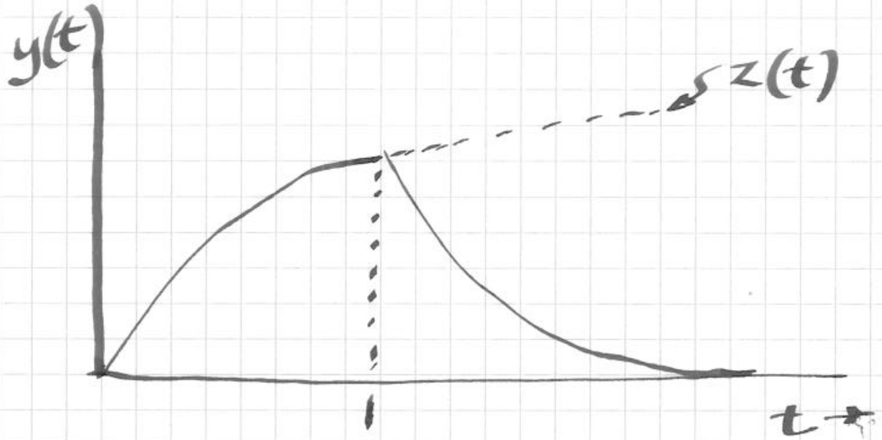
$$e^{-t}u(t) * u(t-1) \Rightarrow z(t-1) \qquad Ⓑ$$

$$= \left[1-e^{-(t-1)}\right]u(t-1)$$

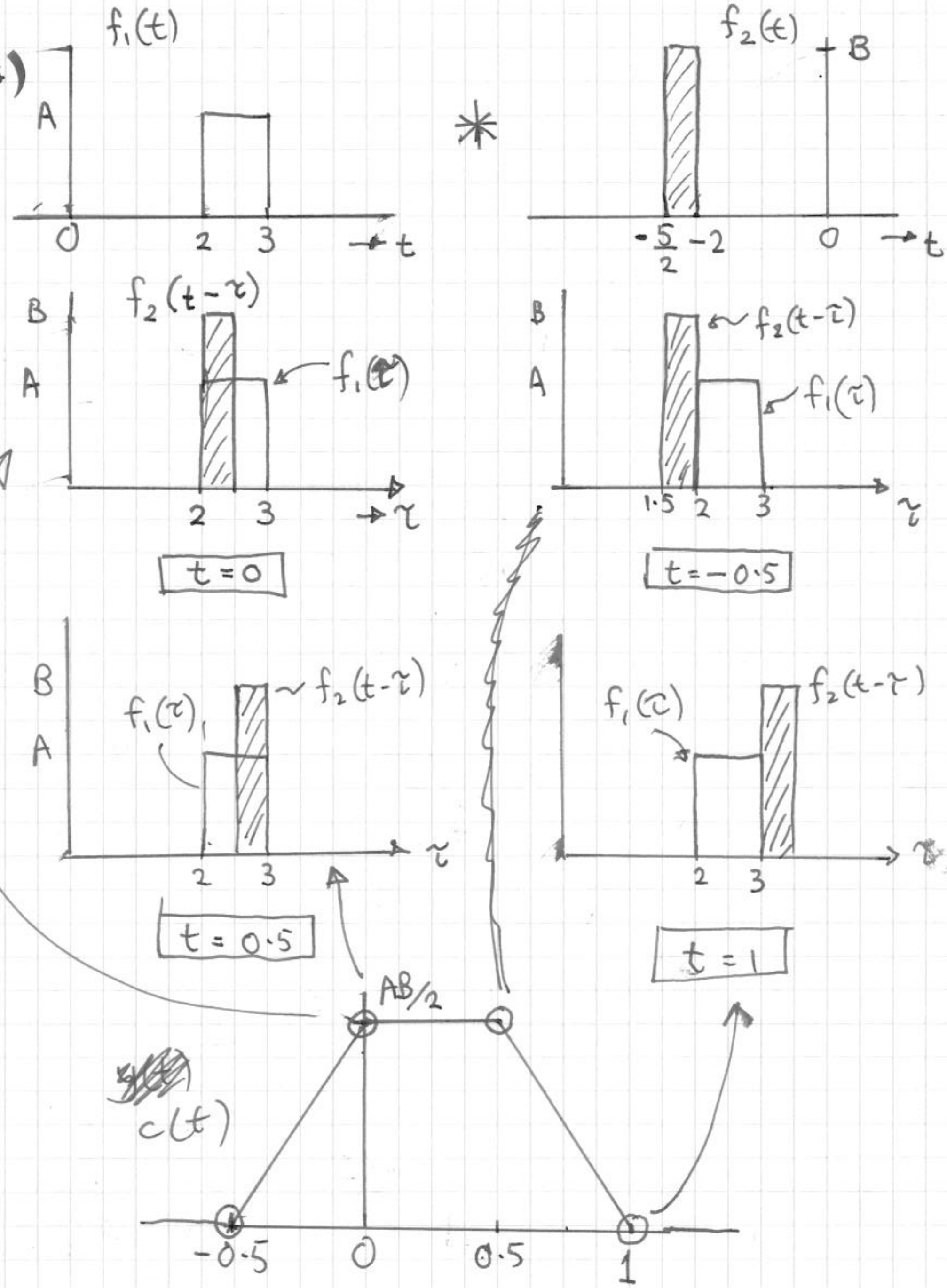
$$\therefore \; y(t) = e^{-t}u(t) * \left[u(t) - u(t-1)\right]$$

$$= (A - B) \qquad \text{superposition}$$

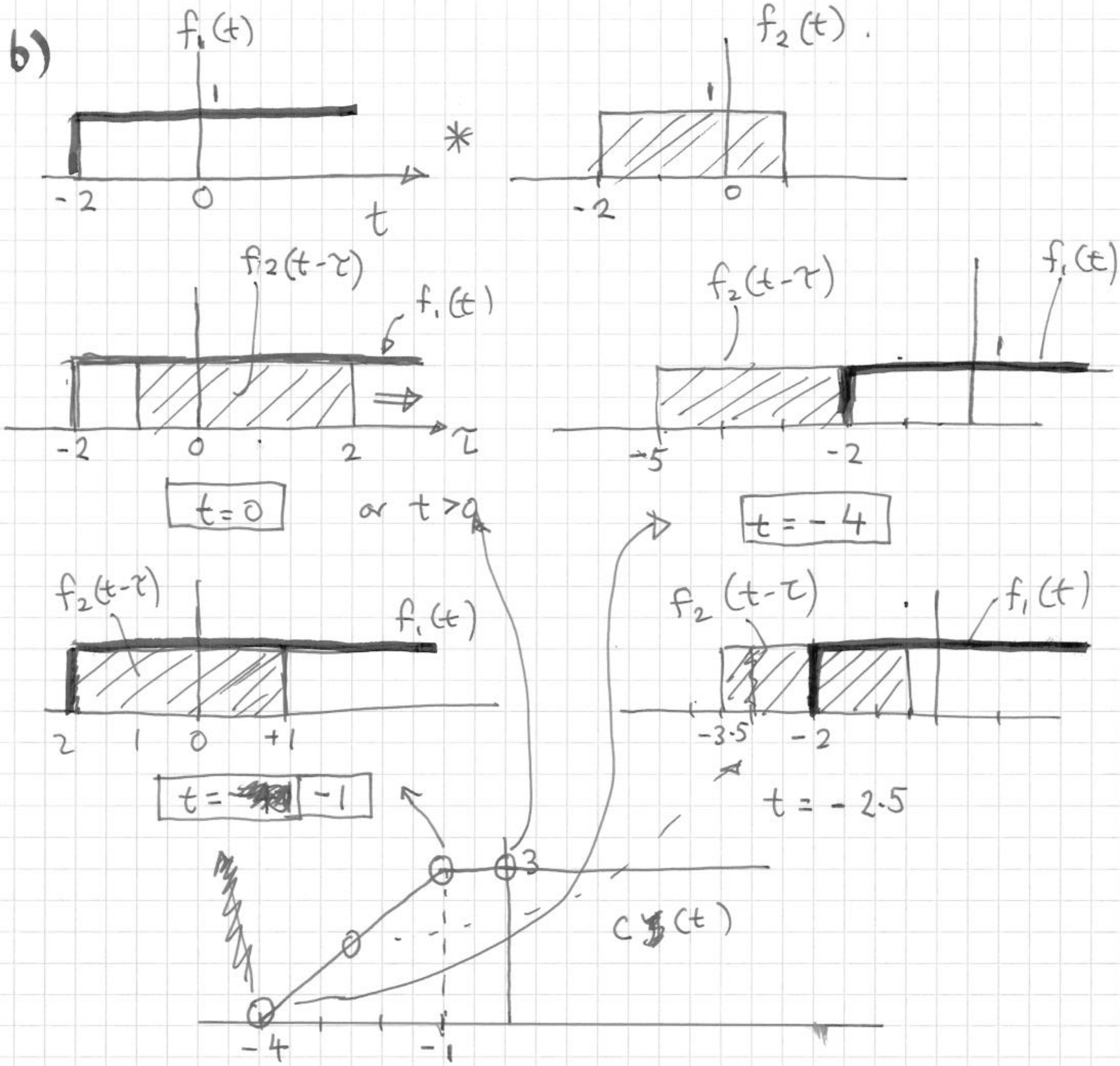
$$= (1-e^{-t})u(t) - \left[1-e^{-(t-1)}\right]u(t-1)$$



Q6 (a)


$f_1(t)$
A
0
2
3
t
*
$f_2(t)$
B
$-\frac{5}{2}$
-2
0
t
$f_2(t-\tau)$
B
A
$f_1(\tau)$
2
3
τ
t = 0
B
$f_2(t-\tau)$
A
$f_1(\tau)$
1.5
2
3
τ
t = −0.5
B
A
$f_1(\tau)$
$f_2(t-\tau)$
2
3
τ
t = 0.5
$f_1(\tau)$
$f_2(t-\tau)$
2
3
t = 1
AB/2
c(t)
−0.5
0
0.5
1

6. b)

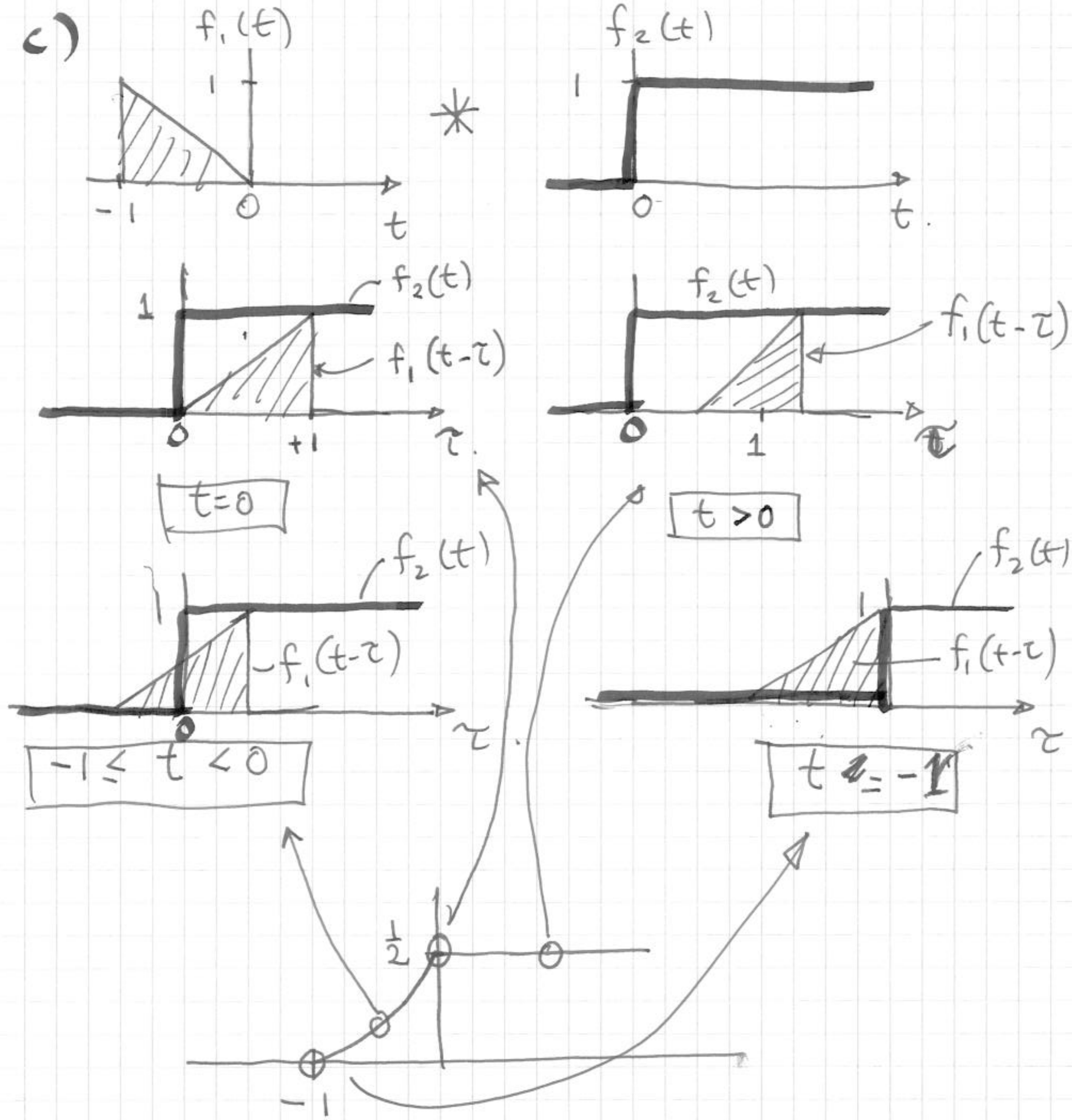


$$c(t) = \int_{-1+t}^{2+t} d\tau = 3 \qquad t \geq -1$$

$$c(t) = \int_{-2}^{2+t} d\tau = t+4 \qquad -1 \geq t \geq -4$$

$$c(t) = 0 \qquad t \leq -4$$

6 c)



Note : Easy to do $\int f_2(\tau)\, f_1(t-\tau)\, d\tau$

$$c(t) = \int_t^{t+1} (\tau - t)\, d\tau = \frac{1}{2} \qquad t \geq 0$$

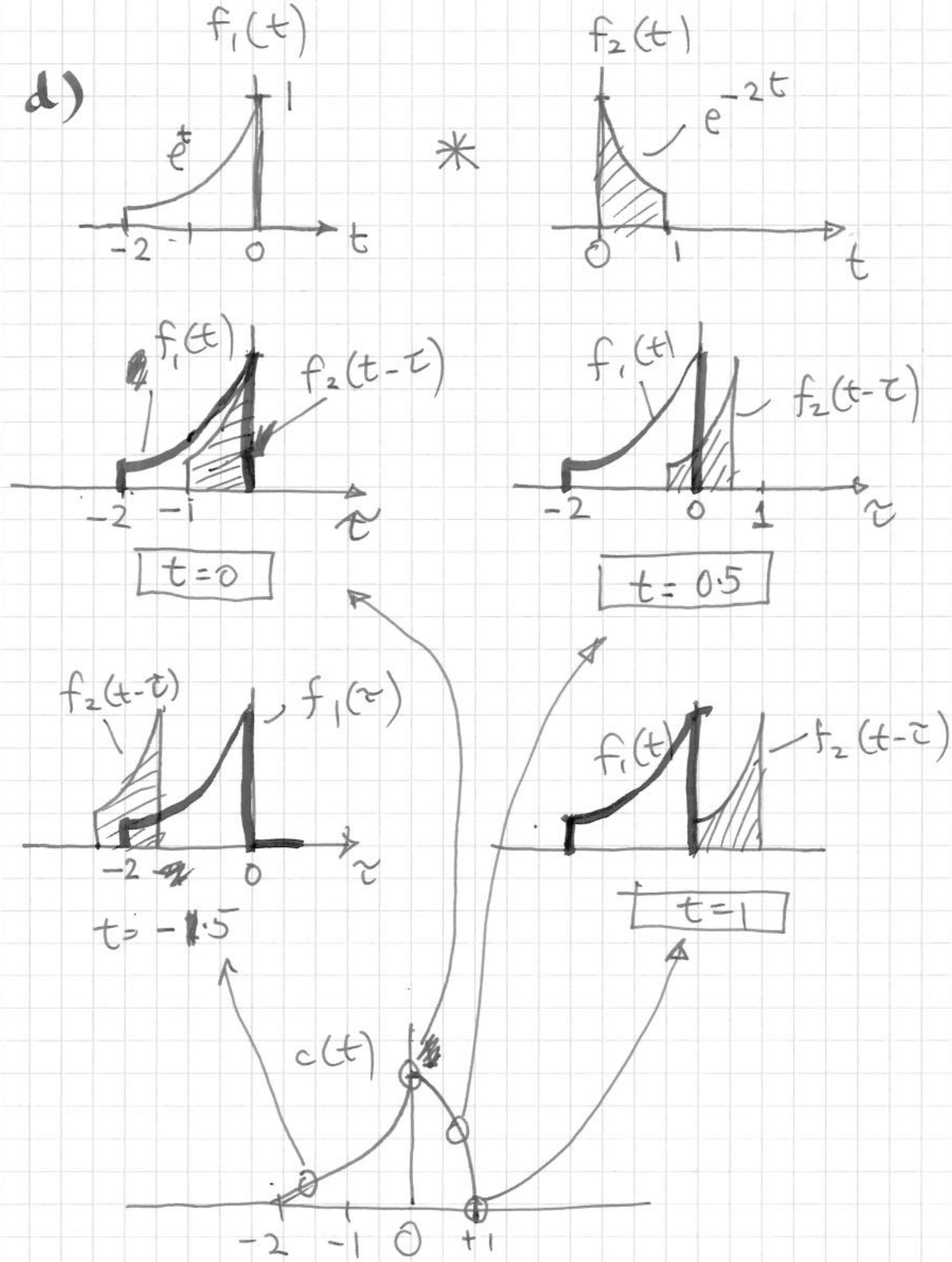
$$c(t) = \int_0^{t+1} (\tau - t)\, d\tau = \frac{1}{2}(1-t^2)$$

$0 \geq t \geq -1$

$c(t) = 0 \qquad t < -1$

S3.8

6 d)



$f_1(t)$
e^{t}
1
-2
-1
0
t
$*$
$f_2(t)$
e^{-2t}
0
1
t
$f_1(t)$
$f_2(t-\tau)$
-2
-1
τ
$t=0$
$f_1(t)$
$f_2(t-\tau)$
-2
0
1
τ
$t=0.5$
$f_2(t-\tau)$
$f_1(\tau)$
-2
0
τ
$t=-1.5$
$f_1(t)$
$f_2(t-\tau)$
$t=1$
$c(t)$
-2
-1
0
+1

S3.9